Math 150, *Lecture Notes- Bonds* **Name____________**

Section 3.7 *Optimization Problems*

Applied Minimum and Maximum Problems

One of the most common applications of calculus involves the determination of minimum and maximum values. Consider how frequently you hear or read terms such as greatest profit, least cost, least time, greatest voltage, optimum size, least size, greatest strength, and greatest distance. Before outlining a general problem-solving strategy for such problems, let's look at an example.

Guidelines for Solving Applied Minimum and Maximum Problems

1. Identify all *given* quantities and quantities *to be determined*. If possible, make a sketch.
objective Equation (Function)
2. Write a **primary equation** for the quantity that is to be maximized or minimized. (A review of several useful formulas from geometry is presented inside the front cover.)
constraints / Drawings / Equation / Reality?
3. Reduce the primary equation to one having a *single independent variable*. This may involve the use of **secondary equations** relating the independent variables of the primary equation.
4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum value by the calculus techniques discussed in Sections 3.1 through 3.4.

When you take a test that covers this material, I will ask you to show me the following:

(i) draw a diagram of the situation and label the diagram

(ii) define your variables

(iii) create a function

(iv) use agraphing utility to graph this function over the appropriate domain

(v) use calculus and algebra to **prove** your result-
that is, differentiate, find critical numbers, use a 1st, or 2nd derivative test

and

(vi) state your answer in a sentence, using correct units

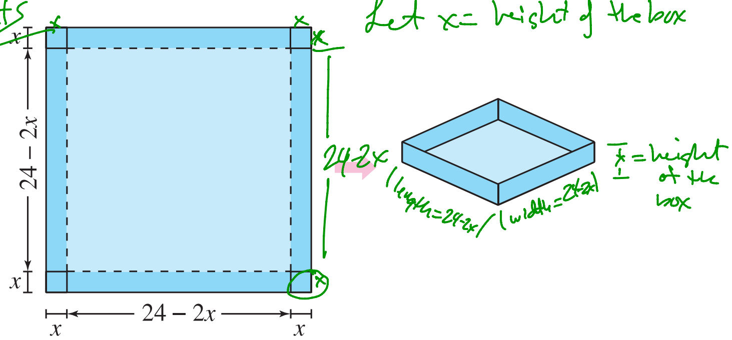
OPTIMIZATION   Objectives   constraints

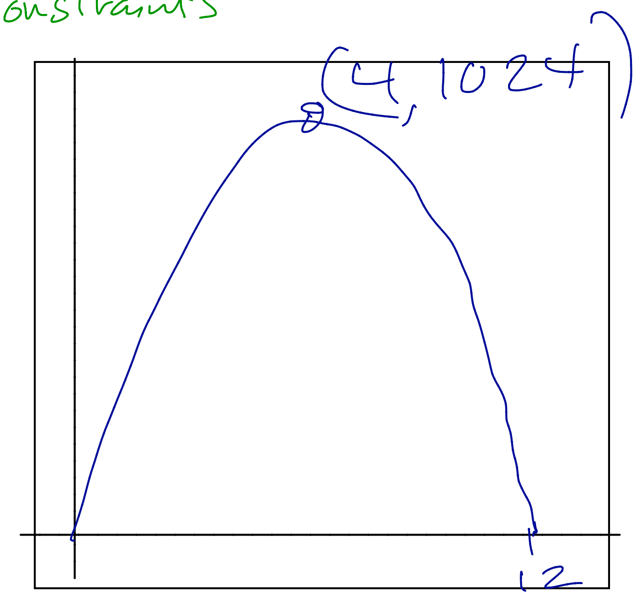
Ex.1

Numerical, Graphical, and Analytic Analysis An open box of maximum volume is to be made from a square piece of material, 24 inches on a side, by cutting equal squares from the the corners and turning up the sides (see figure).

constraints

Let x = height of the box





Objective:

Maximum volume box

Volume of the box = (length)(width)(height)

$V_{box} = (24-2x)(24-2x)(x)$

Function $V_b(x) = (576 - 96x + 4x^2)\,x$

$$\boxed{V_b(x) = 4x^3 - 96x^2 + 576x}$$

$V_b'(x) = 12x^2 - 192x + 576$

$V_b''(x) = 24x - 192$

Find critical Numbers:

(A) $V_b'(x) = 0$

$12x^2 - 192x + 576 = 0$

$\frac{1}{12}\cdot[12x^2 - 192x + 48] = \frac{1}{12}\cdot 0$

$x^2 - 16x + 48 = 0$

$(x-4)(x-12) = 0$

either $x-4=0$, or $x-12=0$

$x=4$, or $x=12$

48: 1,48; 2,24; 3,16; 4,12; 6,8

, (B) $V_b'(x)$ is undefined

Never

Second Derivative Test

cc down   cc up

$V_b''(4) = 24\cdot 4 - 192$

$V_b''(4) = -96 < 0$

Rel. MAX at $x = 4$

$V_b''(12) = 24\cdot 12 - 192$

$V_b''(12) = 288 - 192$

$V_b''(12) = 96 > 0$

?? not MAX

Find the MAX Volume: $V_{box}(4) = [24-2(4)][24-2(4)]\cdot 4$

$= 16\cdot 16\cdot 4$

$= 256\cdot 4$

$= 1024\ in^3$

$400 = 3y + 4x$

Solve for y: $400 - 4x = 3y$

$\frac{400-4x}{3} = y$

$\frac{400-4(50)}{3} = y$

$\frac{200}{3} = y$

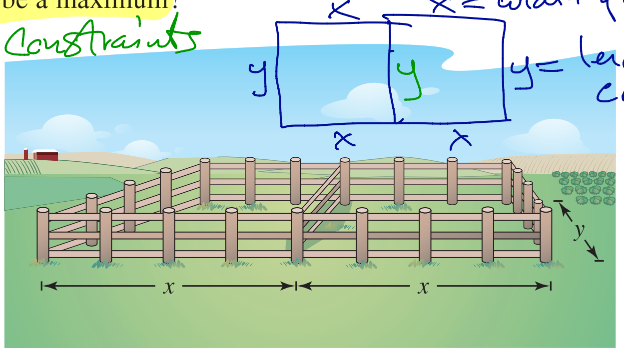
Ex.2

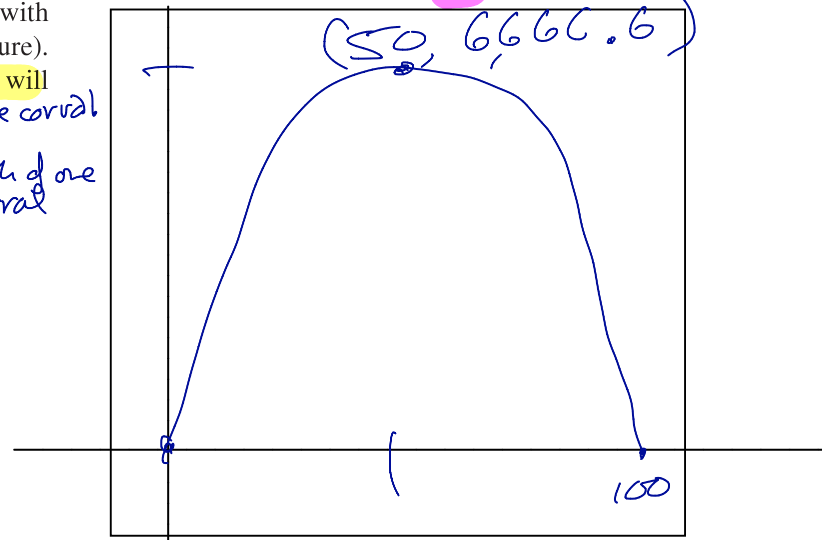
Maximum Area A rancher has 400 feet of fencing with which to enclose two adjacent rectangular corrals (see figure). What dimensions should be used so that the enclosed area will be a maximum?

Constraints

x = width of one corral

y = length of one corral





objective:

Maximum Area of corrals $= (2x)(y)$

$A(x) = (2x)\left(\frac{400-4x}{3}\right)$

$A(x) = \frac{800x}{3} - \frac{8x^2}{3}$

$A'(x) = \frac{800}{3} - \frac{8}{3} \cdot 2x$

$A'(x) = \frac{800}{3} - \frac{16}{3}x$

$A''(x) = \frac{-16}{3} < 0$

Rel. MAX

concave down

Find the critical numbers:

(A) $A'(x) = 0$

$\frac{800}{3} - \frac{16}{3}x = 0$

$\frac{800}{3} = \frac{16x}{3}$

$\frac{800}{16} = \frac{16x}{16}$

$50 = x$

(B) $A(x)$ is undefined

Never

Second Derivative Test

$A''(50) = \frac{-16}{3} < 0$

Find the max area:

$A(50) = [2 \cdot (50)]\left[\frac{400 - 4(50)}{3}\right]$

$= 100\left(\frac{400-200}{3}\right)$

$= \frac{100 \cdot 200}{3}$

$= \frac{20{,}000}{3} = 6{,}666.\bar{6}$ sq. feet

The dimensions of the corrals are 50 feet by 66.6̄ feet.

Ex.3

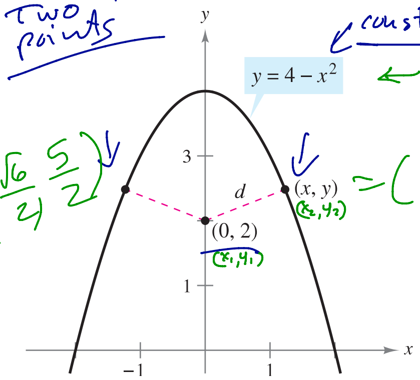
distance $= \sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$

Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$?

Two points

constraints

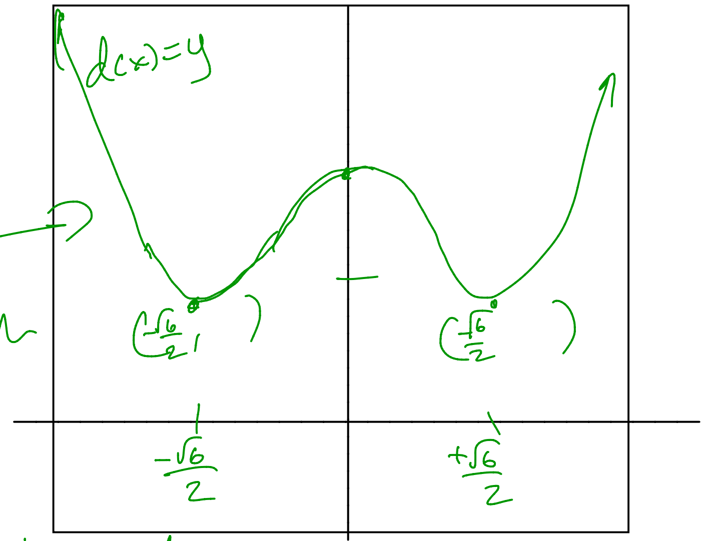
$\leftarrow$ produce a function with one variable



$\left(-\frac{\sqrt{6}}{2}, \frac{5}{2}\right)$ $\qquad = \left(\frac{\sqrt{6}}{2}, \frac{5}{2}\right)$

Figure 3.55

not on the graph



objective – (Function)

minimize the distance between $(0,2)$ and a point on $y = 4 - x^2$

$d = \sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$

$d = \sqrt{(y-2)^2+(x-0)^2}$

$d = \sqrt{(4-x^2-2)^2+x^2}$

$d(x) = \sqrt{(2-x^2)^2+x^2}$

$d(x) = \sqrt{4-4x^2+x^4+x^2}$

$d(x) = \sqrt{x^4-3x^2+4}$

$d^2(x) = x^4-3x^2+4$

$$\boxed{\begin{aligned} d^{2\prime}(x) &= 4x^3-6x \\ d^{2\prime\prime}(x) &= 12x^2-6 \end{aligned}}$$

critical numbers:

(A) $d^{2\prime}(x) = 0$ or (B) $d^{2\prime}$ is undefined never

$4x^3-6x=0$

$2x(2x^2-3)=0$

Either

$2x=0$, or $2x^2-3=0$

$x=0$ $\qquad 2x^2=3$

$x^2 = \frac{3}{2}$

$x = -\frac{\sqrt{6}}{2}$ or $x = +\frac{\sqrt{6}}{2}$

y-coordinates: $y = 4 - x^2$

$y = 4 - \left(\pm\frac{\sqrt{6}}{2}\right)^2$

$y = 4 - \frac{6}{4} = \frac{16-6}{4} = \frac{10}{4} = \frac{5}{2}$

Second Derivative Test:

$d^{2\prime\prime}\left(-\frac{\sqrt{6}}{2}\right) = 12\left(-\frac{\sqrt{6}}{2}\right)^2-6$

$= 12\cdot\left(\frac{6}{4}\right)-6$

$= 18-6$

$= 12 > 0$

Concave up

Rel. Min

$d^{2\prime\prime}\left(+\frac{\sqrt{6}}{2}\right) = 12\left(\frac{+\sqrt{6}}{2}\right)^2-6$

$= 12\cdot\left(\frac{6}{4}\right)-6$

$= 36-6$

$= 18-6$

$= 12 > 0$

Concave up

Rel. Min

Ex.4

A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be $1\frac{1}{2}$ inches, and the margins on the left and right are to be 1 inch (see Figure 3.56). What should the dimensions of the page be so that the least amount of paper is used?

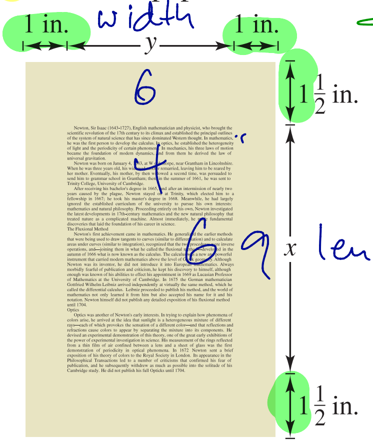
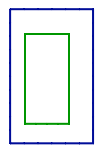


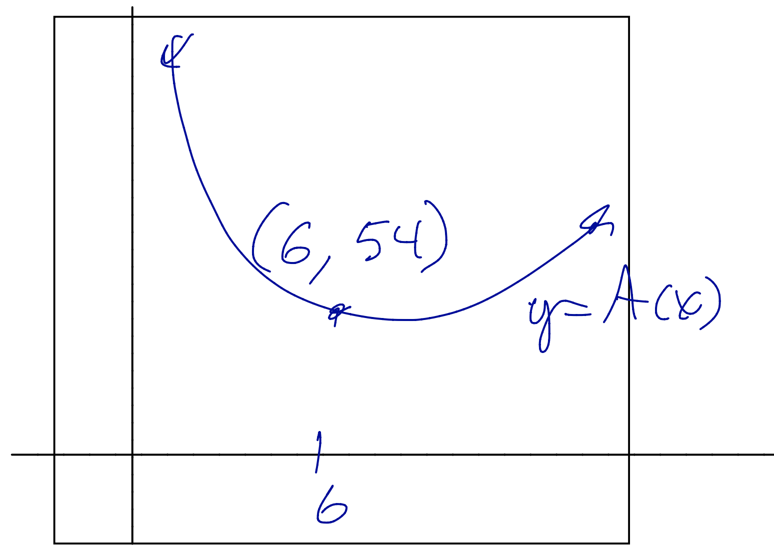
Figure 3.56

constraints:

24 sq inches of print

$$\boxed{24 = x \cdot y}$$





Objective: (Function)

minimize page dimensions – width by length

$\boxed{6 \text{ by } 9}$

width $= y + 2 = 6$

length $= x + 3 = 9$

$24 = xy$

$\frac{24}{x} = y$

$$\text{Area} = (y+2)(x+3)$$

$$A(x) = \left[\frac{24}{x} + 2\right](x+3)$$

$$A(x) = 24 + \frac{72}{x} + 2x + 6$$

$$\boxed{A(x) = 2x + 30 + 72x^{-1}}$$

$$A'(x) = \frac{d}{dx}\left[2x + 30 + 72x^{-1}\right]$$

$$A'(x) = 2 + 72\left[-1 \cdot x^{-2}\right]$$

$$A'(x) = 2 - \frac{72}{x^2}$$

$$A''(x) = \frac{d}{dx}\left(2 - 72x^{-2}\right)$$

$$A''(x) = -72\left(-2x^{-3}\right)$$

$$A''(x) = \frac{144}{x^3}$$

Find critical numbers

(A) $A'(x) = 0$ or (B) $A'(x)$ is undefined

$$2 - \frac{72}{x^2} = 0$$

$$2 = \frac{72}{x^2}$$

$$x^2 = \frac{72}{2}$$

$$x^2 = 36$$

$x = 6$, $x \neq -6$

makes no sense

$x = 0$

$-\frac{72}{x^2}$

Too much white space

Second Derivative Test:

$A''(6) = \frac{144}{6^3} > 0$ Concave Up

Rel. Min

Place the stake 9ft from the shorter post.

Ex.5

Two posts, one 12 feet high and the other 28 feet high, stand 30 feet apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each post. Where should the stake be placed to use the least amount of wire?

Constraints

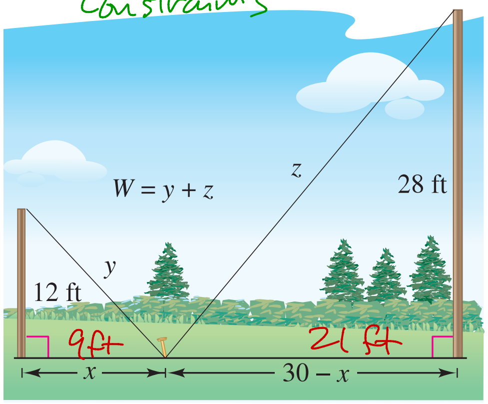
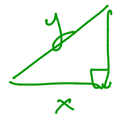
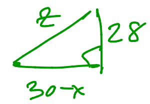


Figure 3.57

$12^2 + x^2 = y^2$



$y = \sqrt{144 + x^2}$

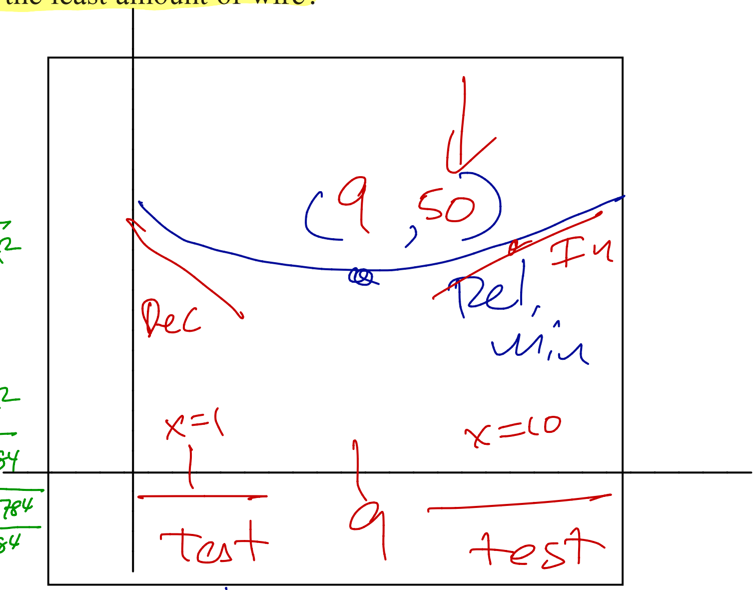


$(30-x)^2 + 28^2 = z^2$

$z = \sqrt{(30-x)^2 + 784}$

$z = \sqrt{900 - 60x + x^2 + 784}$

$z = \sqrt{x^2 - 60x + 1{,}684}$



Objective! (Function)

"least amount of wire"

minimize length of wire

y = wire length used to stay the 12 ft pole

z = wire length used to stay the 28 ft pole

x = distance to the stake

W = amount of wire used

$W = y + z$

$W(x) = \sqrt{144 + x^2} + \sqrt{(30-x)^2 + 784}$

$W(x) = \sqrt{144 + x^2} + \sqrt{x^2 - 60x + 1{,}684}$

Find critical number:

Ⓐ $W'(x) = 0$ or Ⓑ $W'(x)$ is undefined

$x^2 + 144 = 0$ Never , $x^2 - 60x + 1{,}684 = 0$ Never

$$W'(x) = \frac{d}{dx}\left[(144 + x^2)^{1/2}\right] + \frac{d}{dx}\left[(x^2 - 60x + 1{,}684)^{1/2}\right]$$

$$W'(x) = \underbrace{\tfrac{1}{2}(144 + x^2)^{-1/2} \cdot 2x}_{\text{chain rule}} + \underbrace{\tfrac{1}{2}(x^2 - 60x + 1{,}684)^{-1/2} \cdot (2x - 60)}_{\text{chain rule}}$$

$$W'(x) = \frac{x}{\sqrt{144 + x^2}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1684}}$$

critical $x \approx 9$

$$\frac{x}{\sqrt{144 + x^2}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1684}} = 0$$

$$\left(\sqrt{144 + x^2}\right)\left(\sqrt{x^2 - 60x + 1684}\right)\left[\frac{x}{\sqrt{144 + x^2}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1684}}\right] = \left(\sqrt{144 + x^2}\right)\left(\sqrt{x^2 - 60x + 1684}\right)(0)$$

$$x\sqrt{x^2 - 60x + 1684} + (x - 30)\sqrt{144 + x^2} = 0$$

$$\left[(x - 30)\sqrt{144 + x^2}\right]^2 = \left[x\sqrt{x^2 - 60x + 1684}\right]^2$$

$$(x - 30)^2(144 + x^2) = x^2(x^2 - 60x + 1{,}684)$$

$$(x^2 - 60x + 900)(144 + x^2) = x^4 - 60x^3 + 1684x^2$$

$$144x^2 - 8{,}640x + 129{,}600 + x^4 - 60x^3 + 900x^2 = x^4 - 60x^3 + 1{,}684x^2$$

$$\cancel{x^4} - \cancel{60x^3} + 1{,}044x^2 - 8640x + 129{,}600 = \cancel{x^4} - \cancel{60x^3} + 1{,}684x^2$$

$$0 = 640x^2 + 8640x - 129{,}600$$

$$\frac{1}{640}(0) = (640x^2 + 8{,}640x - 129{,}600)\frac{1}{640}$$

$$\frac{1}{5}(0) = (10x^2 + 135x - 2{,}025)\frac{1}{5}$$

$$0 = 2x^2 + 27x - 405$$

$$0 = (2x + 45)(x - 9)$$

$2x + 45 = 0$, or $x - 9 = 0$

$2x = -45$ $\quad$ $x = 9$

$x = \frac{-45}{2}$

$x = -27.5$ ← Not in the Domain

First Derivative Test: $x = 1$ & $x = 10$

$$W'(1) = \frac{1}{\sqrt{144 + (1)^2}} + \frac{1 - 30}{\sqrt{(1)^2 - 60(1) + 1684}}$$

$$W'(1) = \frac{1}{\sqrt{145}} - \frac{29}{\sqrt{1{,}625}} < 0$$

$W'(1) < 0$

W is dec ↘

$$W'(10) = \frac{10}{\sqrt{144 + (10)^2}} - \frac{10 - 30}{\sqrt{(10)^2 - 60(10) + 1684}}$$

$$w'(10) = \frac{10}{\sqrt{244}} - \frac{20}{\sqrt{1{,}184}} > 0$$

$w'(10) > 0$

w is increasing

$$w(9) = \sqrt{144-(9)^2} + \sqrt{(9)^2 - 60(9) + 1684}$$

$$w(9) = \sqrt{144-81} + \sqrt{81 - 540 + 1684}$$

$$= \sqrt{225} + \sqrt{1{,}225}$$

$$= 15 + 35$$

$$w(9) = 50$$

SQUARE $4x$ — circle $4-4x$

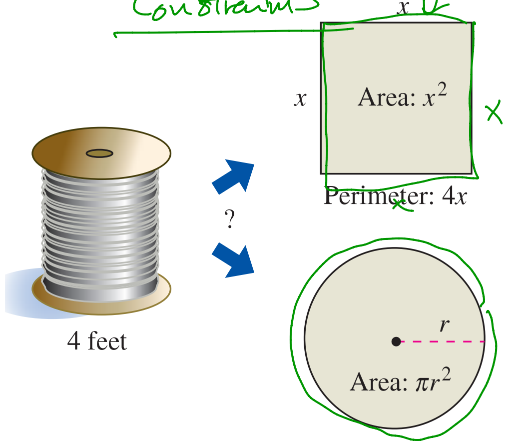
4 feet

Ex.6

Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?

Constraints

$0 \le x \le 4$



Circumference: $2\pi r$

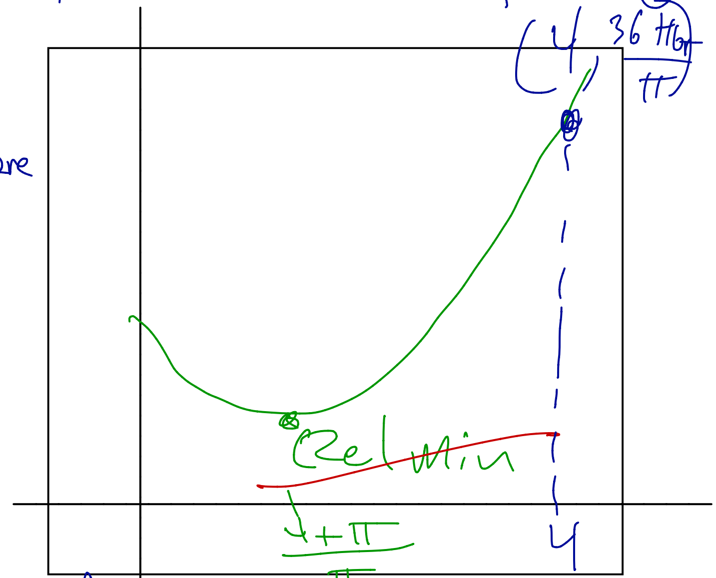
x = side of the square

$4-4x = 2\pi r$

$\frac{4-4x}{2\pi} = r$

$\frac{2-2x}{\pi} = r$

Figure 3.59



objective: (Function)

maximize enclosed Area of a circle & square

$$\text{Area} = x^2 + \pi r^2$$

$$A(x) = x^2 + \pi\left(\frac{2-2x}{\pi}\right)^2$$

$$A(x) = x^2 + \pi\frac{(2-2x)^2}{\pi^2}$$

$$A(x) = \frac{\pi x^2}{\pi} + \frac{4-8x+4x^2}{\pi}$$

$$A(x) = \frac{(4+\pi)x^2 - 8x + 4}{\pi}$$

$$A'(x) = \frac{1}{\pi}\left[2(4+\pi)x' - 8 + 0\right]$$

$$A'(x) = \frac{2(4+\pi)}{\pi}x - \frac{8}{\pi}$$

$$A''(x) = \frac{2(4+\pi)}{\pi}$$

Find Critical numbers

(A) $A'(x) = 0$

(B) $A'(x)$ is undefined never

$$0 = \frac{2(4+\pi)}{\pi}x - \frac{8}{\pi}$$

$$\frac{8}{\pi} = \frac{2}{\pi}(4+\pi)x$$

$$\frac{\pi}{2}\cdot\frac{8}{\pi} = \frac{\pi}{2}\cdot\frac{2}{\pi}(4+\pi)x$$

$$4 = (4+\pi)x$$

$$\frac{4}{4+\pi} = x$$

Use 2nd Derivative Test

$$A''\left(\frac{4}{4+\pi}\right) = \frac{2(4+\pi)}{\pi} > 0$$

concave up

REL min

$x = 0$

$$A(0) = \frac{(4+\pi)(0)^2 - 8(0) + 4}{\pi}$$

$$A(0) = \frac{4}{\pi}$$

$$A(0) \approx 1.27 \text{ feet}^2$$

$x = 4$

$$A(4) = \frac{(4+\pi)(4)^2 - 8(4) + 4}{\pi}$$

$$A(4) = \frac{(4+\pi)16 - 32 + 4}{\pi}$$

$$A(4) = \frac{64 + 16\pi - 28}{\pi}$$

$$A(4) = \frac{36 + 16\pi}{\pi}$$

$$A(4) \approx 27.46 \text{ feet}^2$$